Indian J. Phys. 43, 274–281 (1969)

On the unsteady flow of a viscous incompressible fluid in a channel bounded by two parallel flat plates

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(*Received October* 19, 1968)

In this paper an attempt is made to find the solution of the Navier-Stokes equations for the unsteady flow of a viscous incompressible fluid through a channel bounded by two parallel flat plates under the influence of pressure gradient (i) varying linearly with time, and (ii) decreasing exponentially with time. In the first case it is seen that (a) symmetrical points have the same velocity, and (b) points near the axis of the channel move faster than the points which are far from the axis of the channel.

INTRODUCTION

The present paper consists of two parts. In part A the flow through a channel bounded by two parallel flat plates under pressure gradient varying linearly with time is discussed. An expression for the velocity is obtained in dimensionless form. This consists of two parts, the one varies linearly with the parameter $T = \frac{\nu t}{y_0^2}$ and the other is the transient part of the velocity, which vanishes in the limit as t tends to infinity. It is seen that the contribution of the transient part is insignificant when $T > 1$. It is also observed that for fluid motion with small Reynolds number the transient part of the velocity dies down more quickly than in the case of fluid motion with large Reynolds number.

In part B the flow of a viscous incompressible fluid between two parallel flat plates under exponentially decreasing pressure gradient is studied. An expression for the velocity has been obtained taking

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt},$$

which has been compared with that of Lal's result (1964). Our expression contains some additional terms and the reason for this has been discussed. Our result is in complete agreement with similar results obtained by Ballabh (1959) and Srivastava (1963) where Ballabh has obtained the expression for the velocity by using the method of superposability and Srivastava has discussed the distribution of velocity in a circular pipe under pressure gradient decreasing exponentially with time.

1. Equations of motion

Navier-Stokes equations of motion (Pai 1956) of a viscous incompressible fluid neglecting the external forces are

$$\left.\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\nabla^2 u,\\ \frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\nabla^2 v,\\ \frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\nabla^2 w\end{aligned}\right\} \quad ...(1.1)$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$, and

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad ...(1.2)$$

For the present problem we have

$$\left.\begin{aligned}&u=u(x,y,t),\ v=0,\ w=0,\\ &P=P(x,y,t),\ \frac{\partial}{\partial z}(\ \ )=0.\end{aligned}\right\} \quad ...(1.3)$$

The last equation holds because the motion is two-dimensional.

Furthermore, the equation of continuity (1.2) and the conditions (1.3) give

$$\frac{\partial u}{\partial x}=0 \text{ so that } u=u(y,t). \quad ...(1.4)$$

Substituting equations (1.3) and (1.4) into the equations of motion (1.1), we have

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \quad ...(1.5)$$

and

$$\frac{\partial P}{\partial y}=0 \text{ or } p=p(x,t). \quad ...(1.6)$$

From equations (1.5) and (1.6) we see that $\frac{\partial P}{\partial x}$ must be a constant or a function of time only in the present problem because P is not a function of y, and u is not a function of x.

2. PART A- PRESSURSE GRADIENT VARIES LINEARLY WITH TIME.

Let us assume that

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=a_0+at. \qquad ...(2.1)$$

Equation (1.5) then becomes

$$\frac{\partial u}{\partial t}=a_0+at+\nu\frac{\partial^2 u}{\partial y^2}. \qquad ...(2.2)$$

Let $\bar{u}=\int_0^\infty e^{-st}u\,dt$ be the Laplace transform of u and let u_0 be the initial value of u.

Multiplying equation (2.2) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2\bar{u}}{\partial y^2}-P^2\bar{u}=-\frac{1}{\nu}\left[u_0+\frac{a_0}{s}+\frac{a}{s^2}\right], \qquad ...(2.3)$$

where $P^2=\frac{s}{\nu}$.

We shall now find u_0.

Initially the pressure gradient is a_0 and the motion is steady in the channel.

Hence $\frac{\partial u_0}{\partial t}=0$ and we obtain $\frac{d^2u_0}{dy^2}=-\frac{a_0}{\nu}$. $\qquad ...(2.4)$

The boundary conditions are

$$u_0=0 \text{ when } y=-y_0,$$

and

$$u_0=0 \text{ when } y=y_0.$$

The solution of equation (2.4) under these boundary conditions is

$$u_0=\frac{a_0}{2\nu}(y_0^2-y^2).$$

Substituting this value of u_0 in (2.3), we get

$$\frac{\partial^2\bar{u}}{\partial y^2}-P^2\bar{u}=-\frac{1}{\nu}\left[\frac{a_0}{2\nu}\left(y_0^2-y^2\right)+\frac{a_0}{s}+\frac{a}{s^2}\right]. \qquad ...(2.5)$$

The boundary conditions for $\bar{u}$ are

$$\bar{u}=0 \text{ when } y=-y_0,$$

and

$$\bar{u}=0 \text{ when } y=y_0.$$

The solution of equation (2.5) under these boundary conditions is

$$\bar{u} = \frac{a_0}{2\nu}\left(\frac{y_0^2 - y^2}{s}\right) + \frac{a}{s^3}\left[1 - \frac{\cosh Py}{\cosh Py_0}\right].$$

Now applying Laplace inversion theorem, we get

$$u = \frac{a_0}{2\nu}\left(y_0^2 - y^2\right) + \frac{1}{2\nu}\left(y_0^2 - y^2\right) at - \frac{a}{24\nu^2}\left(5y_0^2 - y_0^2\right)\left(y_0^2 - y^2\right)$$

$$+\frac{64ay_0^4}{\nu^2\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\, e^{-\frac{\nu(2n+1)^2\pi^2 t}{4y_0^2}} \cdot \cos\left[\frac{(2n+1)\pi y}{2y_0}\right]. \quad (2.6)$$

At time $t = 0$, $u = \frac{a_0}{2\nu}\left(y_0^2 - y^2\right)$. Hence from equation (2.6) by putting $t = 0$, we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\cos\left[\frac{(2n+1)\pi y}{2y_0}\right] = \frac{\pi^5}{1536}\left(5 - \frac{y^2}{y_0^2}\right)\left(1 - \frac{y^2}{y_0^2}\right).$$

Writing $\frac{y}{y_0} = r$ so that $|r|$ is less than 1, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\cos\left[\frac{(2n+1)\pi}{2}r\right] = \frac{\pi}{1536}\left(5 - r^2\right)\left(1 - r^2\right).$$

Putting $r = 0$, we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

Now we make equation (2.6) dimensionless by introducing

$$U = \frac{u}{U_0},\ \frac{y}{y_0} = r,\ T = \frac{\nu t}{y_0^2},$$

where U_0 is a characteristic velocity.

We then get

$$U = b_0(1 - r^2) + bT(1 - r^2) - \frac{b}{12}(5 - r^2)(1 - r^2)$$

$$+\frac{128b}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\cdot e^{-\frac{(2n+1)^2\pi^2}{4}T}\cos\left[\frac{(n+1)\pi}{2}r\right] \quad \ldots(2.7)$$

where $b_0 = \frac{a_0 y_0^2}{2\nu U_0}$ and $b = \frac{a y_0^4}{2\nu^2 U_0}$ are clearly dimensionless numbers.

We now take $U = U_1 + U_2$, where $U_1 = b_o(1 - r^2) + bT(1 - r^2) - \frac{b}{12}(5 - r^2)(1 - r^2)$; and $U_2 = \frac{128b}{\pi^5}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} \cdot e^{-\frac{(2n+1)^2\pi^2}{4}T} \cdot \cos\left[\frac{(2n+1)\pi}{2} r\right]$.

The values of U for different values of r and T have been tabulated.

TABLE 1. $b_o = 2$, $b = 1$

r \ T	0 01	0.1	0.2	0.3	0.4	0.8	1.0
0.0	2.0019	2.0256	2 0422	2.0874	2.1435	2.4459	2.6213
0.3	1.8220	1.8438	1.8603	1.9025	1.9544	2.2314	2.3914
0.6	1.2800	1.2979	1.3127	1.3445	1.3827	1.5813	1.6948
0.9	0.3800	0.3862	0.3921	0.4026	0.4147	0.4754	0.5096

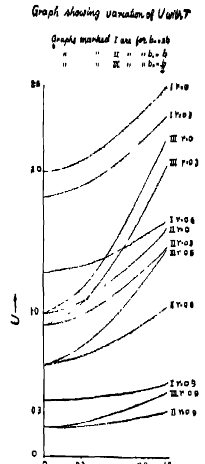
TABLE 2. $b_o = 1$, $b = 1$

r \ T	0.01	0.1	0.2	0.3	0.4	0.8	1.0
0.0	1.0019	1.0256	1 0422	1.0874	1.1435	1.4459	1.6213
0.3	0.9120	0.9338	0.9503	0.9925	1.0444	1.3214	1.4814
0.6	0 6400	0.6579	0.6727	0.7045	0.7427	0.9413	1.0548
0.9	0.1900	0 1962	0.2021	0.2126	0.2247	0.2854	0.3196

TABLE 3. $b_o=1$, $b=2$

r \ T	0.01	0.1	0.2	0 3	0.4	- 0.8	1.0
0.0	1.0038	1.0512	1.0844	1.1748	1.2870	1.8918	2.2426
0.3	0.9140	0.9576	0.9906	1.0750	1.1788	1.7328	2.0528
0.6	0.6400	0.6758	0.7054	0.7690	0.8454	1.2426	1.4696
0.9	0.1900	0.2024	0.2142	0.2352	0.2594	0.3808	0.4492

The graphs for fixed r ($r = 0, +0.3, +0.6, +0.9$) showing the variation of U with the parameter T have been drawn in three cases $b_o=1, b=1$, $b_o = 1, b = 2$, and $b_o = 2, b=1$ in the range $T = 0$ to $T = 1$. The graphs for negative values of r have not been drawn because of the fact that the velocity will not change whether r is negative or positive. It means that the symmetrical points have the same velocity.



The graphs beyond $T=1$ have not been drawn because U_2 is very small compared to U_1 when $T>1$, hence the transient part is insignificant and U varies linearly with T in this range. From the graphs and tables of values it is observed that U increases with T for fixed r. It is also seen that for any T, U decreases with the increases of r and it is maximum when $r=0$. It means that the points near the axis of the channel move faster than the points which are far from the axis of the channel.

It can also be easily seen that for fluids with small Reynold number, the transient part of U becomes insignificant after an interval of time which is shorter than the time required in the case of fluids with large Reyonlds number.

3. PART B. PRESSURE GRADIENT DECREASES EXPONENTIALLY WITH TIME.

We take

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=a_0+\sum_{m=1}^{\infty}a_m e^{-mt}. \qquad \ldots(3.1)$$

Equation (1.5) then becomes

$$\frac{\partial u}{\partial t}=a_0+\sum_{m=1}^{\infty}a_m e^{-mt}+\nu\frac{\partial^2 u}{\partial y^2}\ . \qquad \ldots(3.2)$$

Let $\bar{u}=\int_0^{\infty}e^{-st}\,u\,dt$ be the Laplace transform of u and let u_0 be the initial value of u, Multiplying equation (3.2) by e^{-st} and integrating betwen the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{u}}{\partial y^2} - P^2 \bar{u} = -\frac{1}{\nu}\left[u_0 + \frac{a_0}{s} + \sum_{m=1}^{\infty} \frac{a_m}{(s+m)} \right], \qquad \dots (3.3)$$

where $P^2 = \frac{s}{\nu}$.

Here again $u_0 = \frac{a_0}{2\nu}(y_0^2 - y^2)$.

The solution of the equation (3.3) under the boundary conditions

$\bar{u} = 0$ when $y = -y_0$,

and $\bar{u} = 0$ when $y = y_0$

is

$$\bar{u} = \frac{a_0}{2\nu}\left(\frac{y_0^2 - y^2}{s}\right) + \left[1 - \frac{\cosh Py}{\cosh Py_0}\right] \sum_{m=1}^{\infty} \frac{a_m}{s(s+m)} \qquad \dots (3.4)$$

Now applying Laplace inversion theorem, we get

$$u = \frac{a_0}{2\nu}\left(y_0^2 - y^2\right) - \sum_{m=1}^{\infty} \frac{a_m}{m} \left[1 - \frac{\cos\left\{\left(\frac{m}{\nu}\right)^{\frac{1}{2}} y\right\}}{\cos\left\{\left(\frac{m}{\nu}\right)^{\frac{1}{2}} y\right\}} \right] e^{-mt}$$

$$+ \frac{4}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a_m}{(2n+1)\left[m - \frac{\nu(2n+1)^2\pi^2}{4y_0^2} \right]} \cdot e^{-\frac{\nu(2n+1)^2\pi^2 t}{4y_0^2}} .$$

$$\cos\left[\frac{(2n+1)\pi y}{2y_0}\right] = U'_1 + U'_2 + U'_3 \qquad \dots (3.5)$$

This expression for the velocity does not agree with Lal's result. His expression does not contain U'_3. The difference lies in the fact that Lal has assumed the form of u as

$$u = u_0 + \sum_{m=1}^{\infty} u_m e^{-mt},$$

where u_0 and u_m are fuctions of r only. Naturally then the part U'_3 will be absent in his expression. But the expression (3.5) is in full agreement with Ballabh's result where he has obtained the expression for the velocity by the method of superposability. The expression (3.5) for the velocity is also similar to an expression obtained by Srivastava where he has found the velocity of an incompressible fluid in a circular pipe under exponentially decreasing pressure gradient.

Hence $U'_1+U'_2+U'_3$ is a more general solution of (3.2) and this is confirmed by Laplace Transform method used in the present paper.

The author is indebted to Dr. P. L. Bhatnagar for kind guidance in the preparation of this paper.

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